Response of the biological pump to perturbations in the iron supply: Global teleconnections diagnosed using an inverse model of the coupled phosphorus-silicon-iron nutrient cycles.

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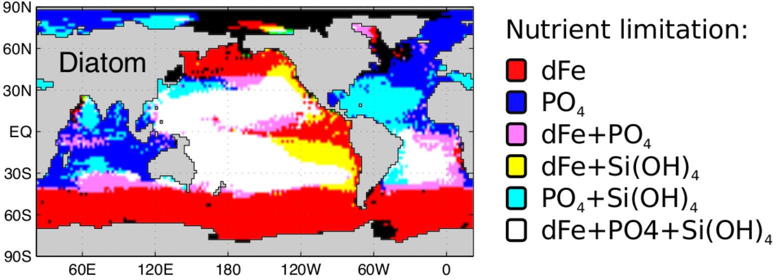
Motivation

- Iron is a key limiting nutrient in HNLC regions [*Boyd et al.*, 2007].



- **Question**: How do the global nutrient cycles respond to perturbations in the iron supply?
  - High-latitude control on biological productivity (dFe not modeled) [*Sarmiento et al.*, 2004; *Primeau et al.*, 2013; *Holzer and Primeau*, 2013]
  - Iron input perturbations using forward models [e.g., *Dutkiewicz et al.*, 2005; *Nickelsen and Oschlies*, 2015]
- We built a data-constrained, inverse model that couples P, Si, and Fe.

Model: Tracer Equations

$$
\begin{aligned}
\mathcal{T}\chi_{\mathrm{P}} &= \textstyle\sum_c(\mathcal{S}_c^{\mathrm{P}} - 1)U_c \;-\; \gamma_g(\chi_{\mathrm{P}} - \overline{\chi}_{\mathrm{P}}^{\mathrm{obs}}) \\
\mathcal{T}\chi_{\mathrm{Si}} &= (\mathcal{S}^{\mathrm{Si}} - 1)R^{\mathrm{Si:P}}U_{\mathrm{dia}} \;-\; \gamma_g(\chi_{\mathrm{Si}} - \overline{\chi}_{\mathrm{Si}}^{\mathrm{obs}}) \\
\mathcal{T}\chi_{\mathrm{Fe}} &= \textstyle\sum_c(\mathcal{S}_c^{\mathrm{Fe}} - 1)R_c^{\mathrm{Fe:P}}U_c \;+\; s_{\mathrm{A}} + s_{\mathrm{S}} + s_{\mathrm{H}} \\
&\qquad + \;(\mathcal{S}^{\mathrm{sPOP}} - 1)J_{\mathrm{POP}} + (\mathcal{S}^{\mathrm{sbSi}} - 1)J_{\mathrm{bSi}} - J_{\mathrm{dst}}
\end{aligned}
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$$\mathcal{T}\chi_{\mathrm{Fe}} = \textstyle\sum_c (\mathcal{S}_c^{\mathrm{Fe}} - 1) R_c^{\mathrm{Fe:P}} U_c \; + \; s_{\mathrm{A}} + s_{\mathrm{S}} + s_{\mathrm{H}} \; + \; (\mathcal{S}^{\mathrm{sPOP}} - 1) J_{\mathrm{POP}} + (\mathcal{S}^{\mathrm{sbSi}} - 1) J_{\mathrm{bSi}} - J_{\mathrm{dst}}$$

$\mathcal{T} =$ advective-diffusive transport
(data-assimilated [*Primeau et al.*, 2013])

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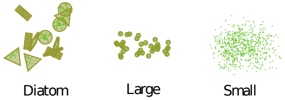

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Matsumoto et al., [2013]

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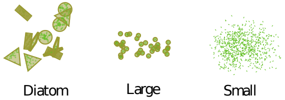

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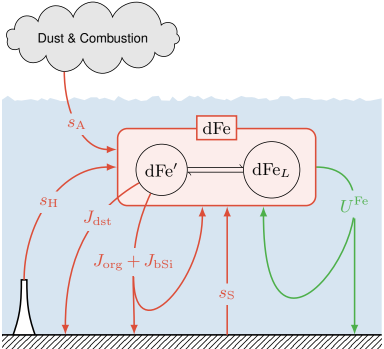

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3 sources of iron:

- Aeolian, s_{A}
- Sedimentary, s_{S}
- Hydrothermal, s_{H}

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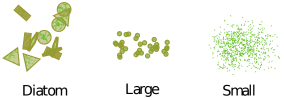

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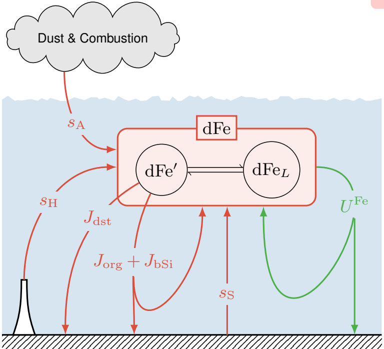

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3 sources of iron:

- Aeolian, s_{A}
- Sedimentary, s_{S}
- Hydrothermal, s_{H}

3 sinks:

- POP, J_{POP}
- Opal, J_{bSi}
- Dust, J_{dst}

Model: Nutrient Uptake and Limitation

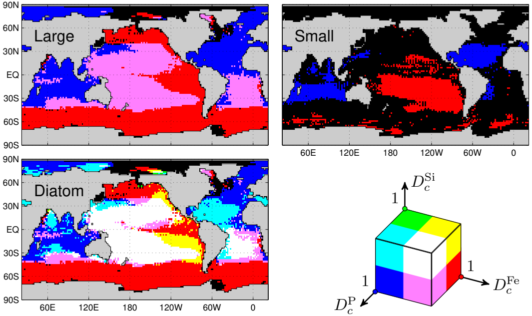
- PO_4-uptake function of temperature, light and nutrient availability

$$U_c = \mu_c \, p_c = \frac{p_c^{\max}}{\tau_c} \, e^{\kappa T} \left(F_I \;\; F_{\mathrm{N},c} \right)^2$$

(Derived from a logistic equation [*Dunne et al.*, 2005])

- Nutrient limitation: product of Monod factors for each nutrient i

$$F_{\mathrm{N},c} = \prod_i \frac{\chi_i}{\chi_i + k_c^i} \equiv \prod_i (1 - D_c^i)$$



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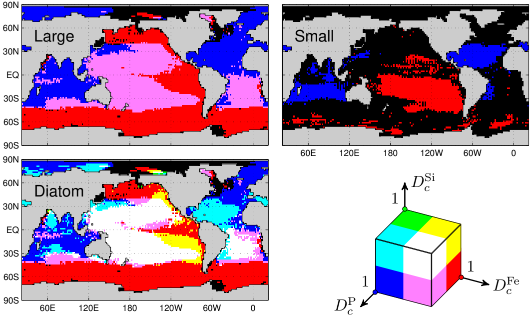
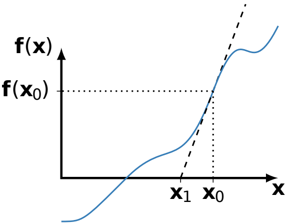


Parameter optimization: Inverse Modelling

- 34 BGC parameters are adjusted to minimize the mismatch with observed nutrient and phytplankton concentrations, χ_i^{obs} and p_c^{obs}:

$$\text{cost} = \sum_i \omega_i \int dV (\chi_i^{\mathsf{mod}} - \chi_i^{\mathsf{obs}})^2 + \sum_c \omega_c \int dV (p_c^{\mathsf{mod}} - p_c^{\mathsf{obs}})^2.$$

- Parameter optimization requires to solve the tracer equations thousands of times!
- Solution: we use a Newton Solver [e.g., *Kelley*, 2003] to solve the discretized nonlinear equations (∼600 000 equations and unknowns) which converges typically in ∼10 iterations.



No spin-up ⇒ fast!

Results: Estimates of the current state of the ocean

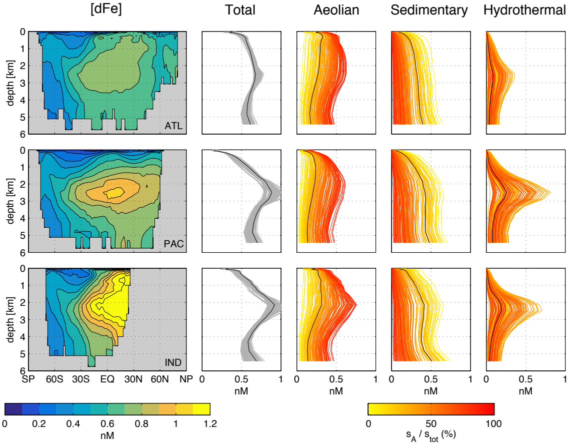
Iron sources in literature:
2 orders of magnitude range
[*Tagliabue et al.*, 2015]

All consistent with observations:

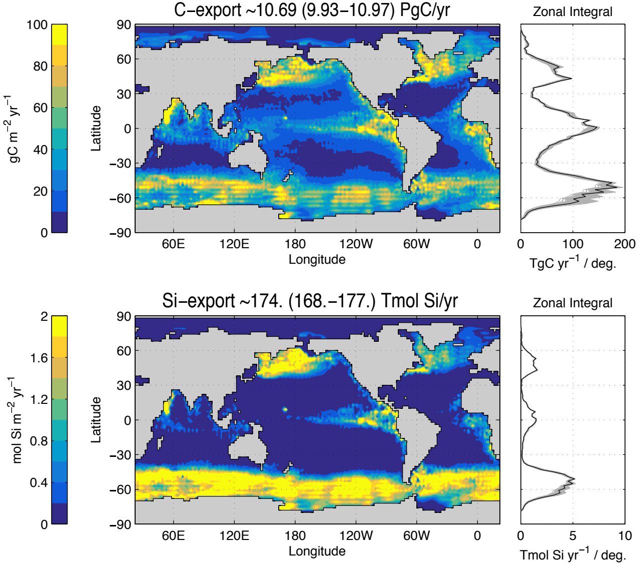
- PO_4: RMS $\sim$ 0.1 mmol/m^3 (5 %)
- $Si(OH)_4$: RMS $\sim$ 10 mmol/m^3 (12 %)
- dFe: RMS $\sim$ 0.28 nM (43 %)

We chose a range
of iron sources:

- $s_A \sim$ 0–15 GmolFe/yr
- $s_S \sim$ 0–12 GmolFe/yr
- $s_H \sim$ 0–3 GmolFe/yr

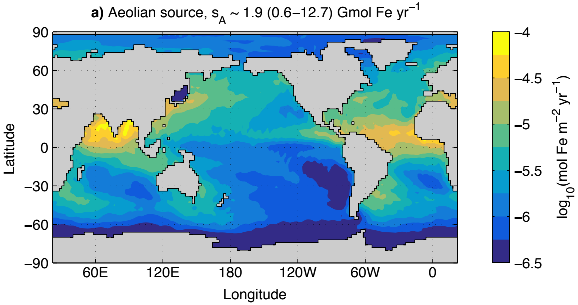


Results: Carbon and Opal Export Productions are well constrained



Perturbations of the Tropical Aeolian Iron Supply

Map of aeolian iron source pattern [*Luo et al.*, 2008]:



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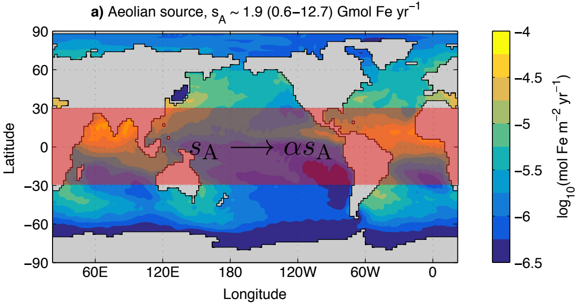
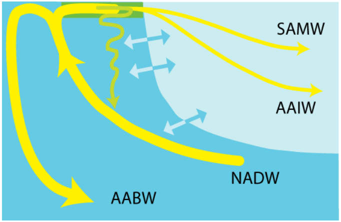


Between 30°S and 30°N, we multiply the aeolian iron source by α, ranging from 0–100.

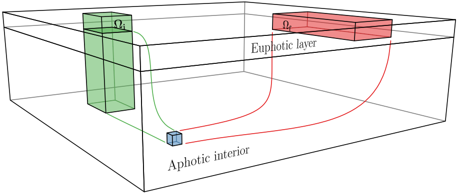
Southern Ocean Trapping and Path Density Diagnostic

- Southern Ocean ($< 38\,^\circ$S) nutrient trapping [*Holzer et al.*, 2014]:
  - Preformed nutrients are "leaked" from the SO via mode and intermediate waters
  - Regenerated nutrients are remineralized within upwelling circumpolar deepwater (and trapped)

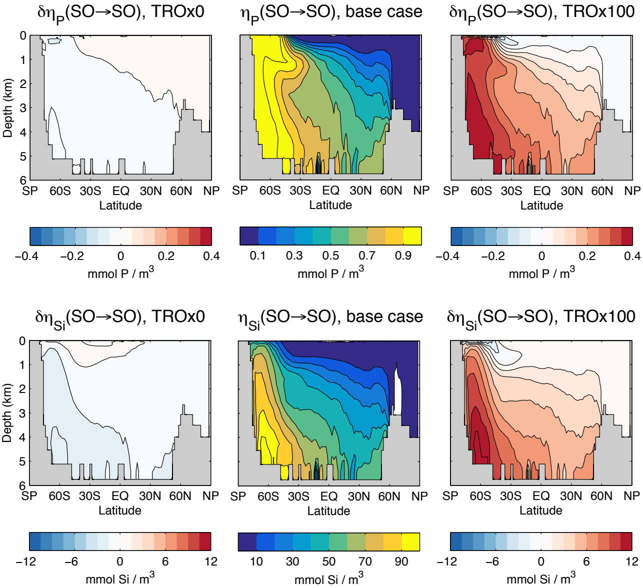


- Path Density from uptake in region Ω_i to uptake in region Ω_f:
  - Concentration of nutrient i last taken up in SO: $g_i^{\downarrow}(\boldsymbol{r}|\mathrm{SO})$
  - Fraction of nutrient to be next taken up in SO: $f_i^{\uparrow}(\boldsymbol{r}|\mathrm{SO})$
  - Path density of nutrient "trapped" in SO$\rightarrow$SO transit:

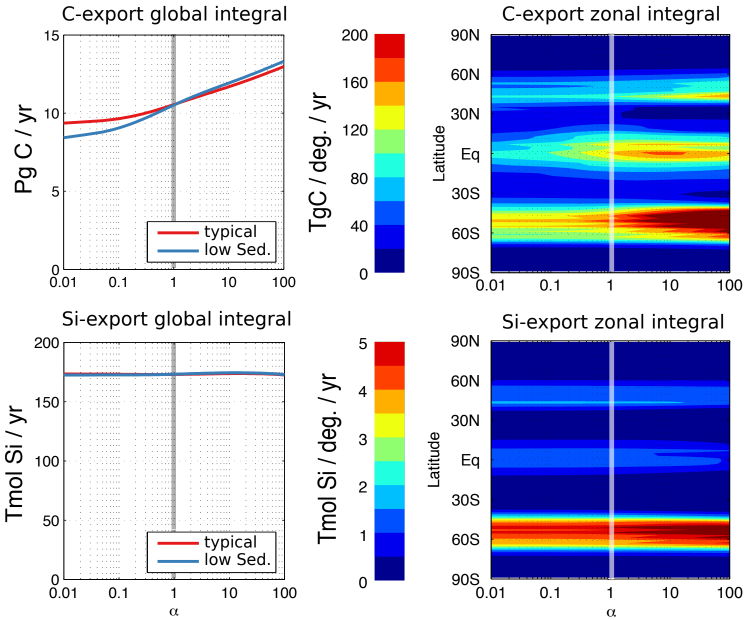
$$\eta_i(\boldsymbol{r}|\mathrm{SO} \rightarrow \mathrm{SO}) = g_i^{\downarrow}(\boldsymbol{r}|\mathrm{SO}) f^{\uparrow}(\boldsymbol{r}|\mathrm{SO})$$



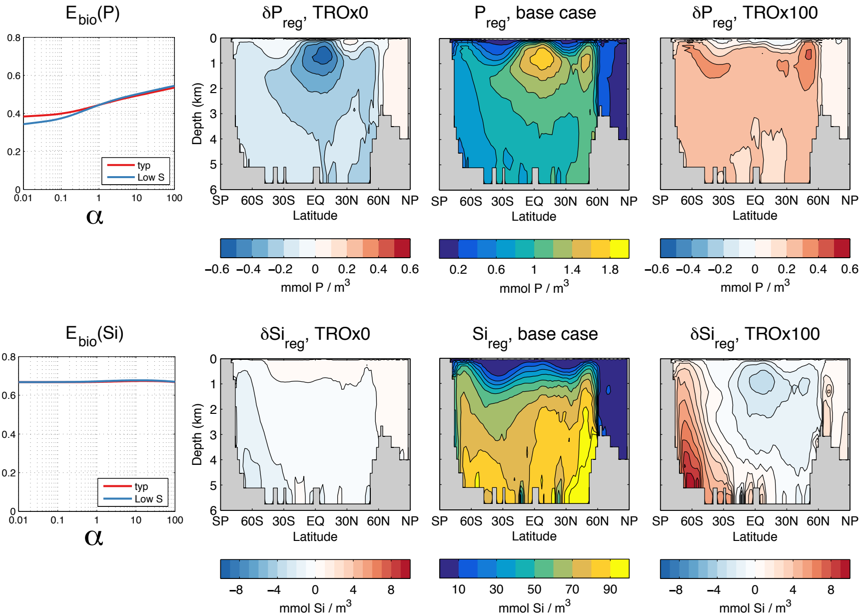
Perturbations: Increased iron input ⇒ Increased SO Trapping



Perturbations: Export Production Response



Perturbations: Regenerated Nutrient Response



Summary and Conclusions

- We have an efficient model coupling the P, Si, and Fe cycles, embedded in a data-assimilated steady circulation:
  - Computational efficiency allows for optimization of BGC parameters (inverse modelling) and for numerous perturbation experiments.
  - The current sparse dFe observations are consistent with a large range of iron source strengths.
- Global response to tropical perturbations of the aeolian iron input:
  - The initial state of the unperturbed iron cycle (e.g., low sedimentary source) determines the sensitivity of nutrient cycles to perturbations.
  - Tropical perturbations have a strong high-latitude influence, particularly for Southern Ocean productivity and nutrient trapping.
  - Increased tropical aeolian Fe input plugs the Southern Ocean leak of the biological pump.
  - The Si-cycle is less sensisitive to iron perturbations than the P-cycle because changes of the Si:P uptake ratio compensates changes in export production.

Discretized PDE

- The tracer equation is $\partial_t \mathbf{x} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x}$ represents the (3-dimensional) concentration fields of the 3 nutrients, rearranged into a single column vector of size $n \sim 600{,}000$

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- The function $\mathbf{f}$ combines the advective-eddy-diffusive transport

$$\mathbf{f}(\mathbf{x}) = \overbrace{-\begin{bmatrix} \mathbf{T} & & \\ & \mathbf{T} & \\ & & \mathbf{T} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{p} \\ \mathbf{s} \\ \mathbf{f} \end{bmatrix}}_{= \mathbf{x}}}^{\text{transport (linear)}}$$

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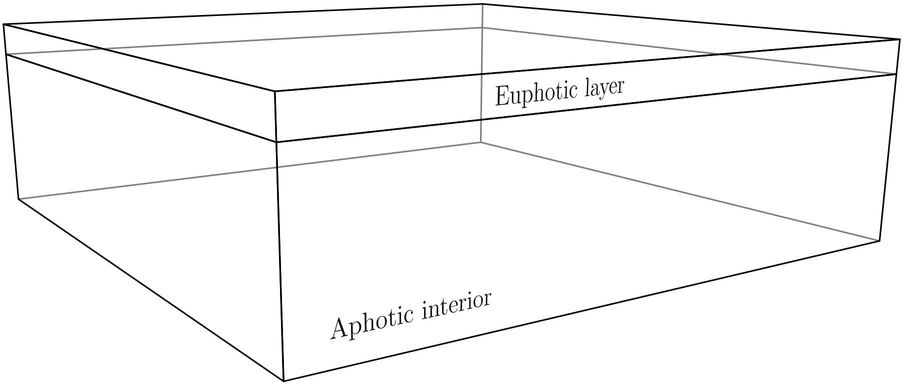
$$\begin{bmatrix} \mathbf{p} \\ \mathbf{s} \\ \mathbf{f} \end{bmatrix} = \mathbf{x}$$

- We solve the steady state equation $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ using Newton's Method, i.e. we solve 600,000 equations in 600,000 unknowns!

Path densities: Definition



- Regenerated nutrient = remineralized at depth that has not yet reemerged in the euphotic zone
- The path density $\langle \boldsymbol{\eta}_{\text{reg}}(\mathbf{r}) \rangle$ is the concentration of regenerated nutrients at $\mathbf{r}$ last taken up Ω_i that is destined to reemergence in Ω_f

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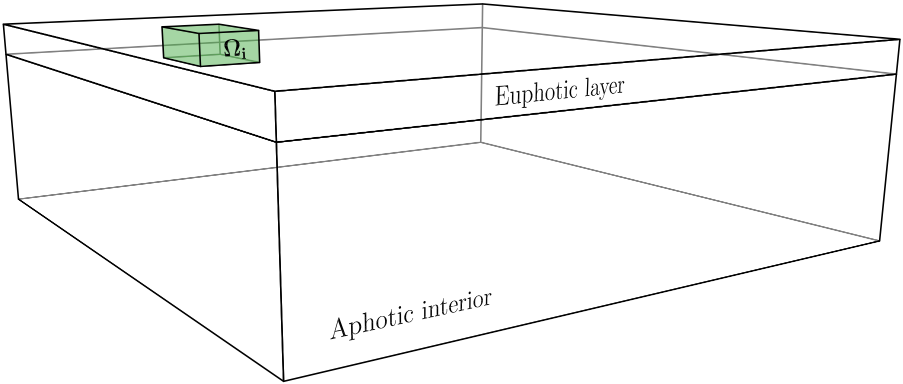

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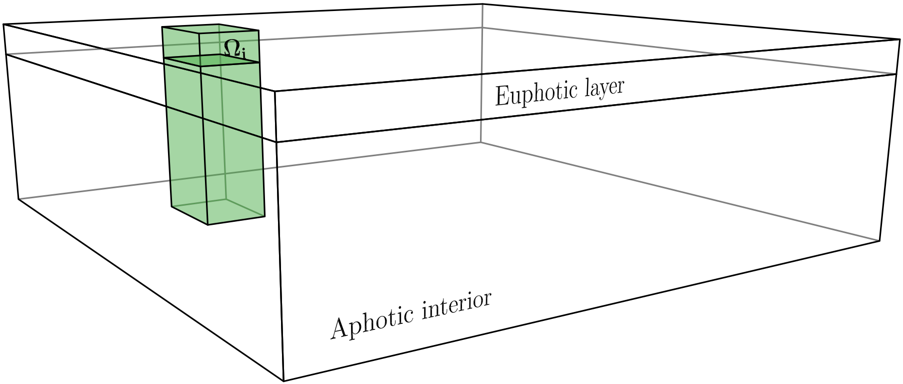

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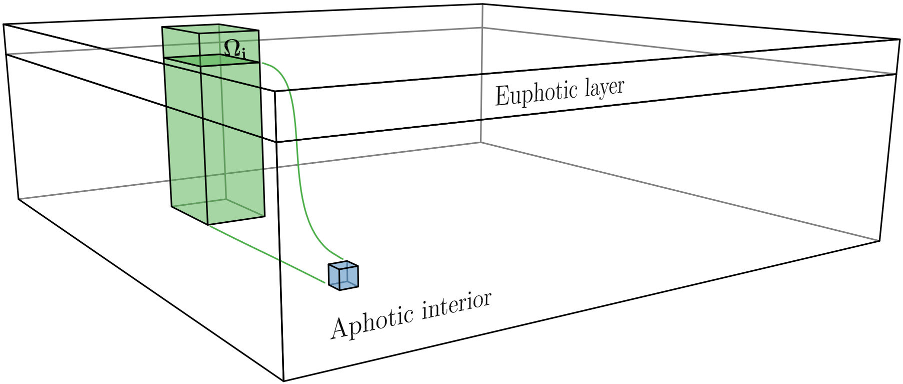

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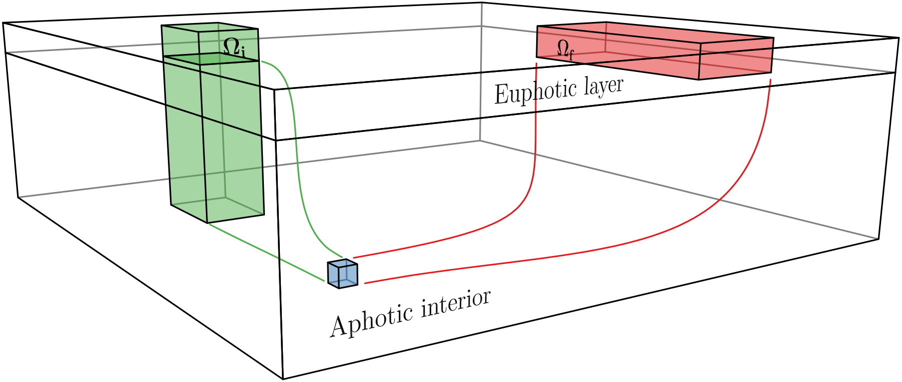

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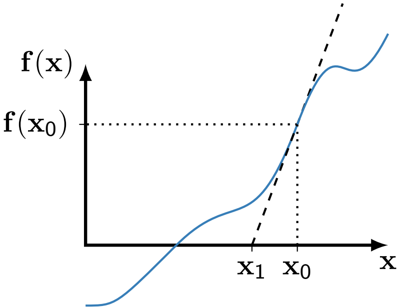
Newton PDE solution

- steady state: $\partial_t \mathbf{x} = \mathbf{f}(\mathbf{x}) = \mathbf{0}$
- use Newton's Method (generalized zero search)
  linear approximation:

$$\mathbf{f}(\mathbf{x}_1) = \mathbf{f}(\mathbf{x}_0) + \mathbf{Df}(\mathbf{x}_0)\,(\mathbf{x}_1 - \mathbf{x}_0) + o\,(\|\mathbf{x}_1 - \mathbf{x}_0\|)$$



where $\mathbf{Df}$ is the Jacobian,
a $n \times n$ sparse matrix
where $n \sim$ 600,000!

To get $\mathbf{f}(\mathbf{x}_1) \sim \mathbf{0}$,
we take

$$\mathbf{x}_1 = \mathbf{x}_0 - \mathbf{Df}(\mathbf{x}_0)^{-1}\mathbf{f}(\mathbf{x}_0)$$

Kelley, 2003

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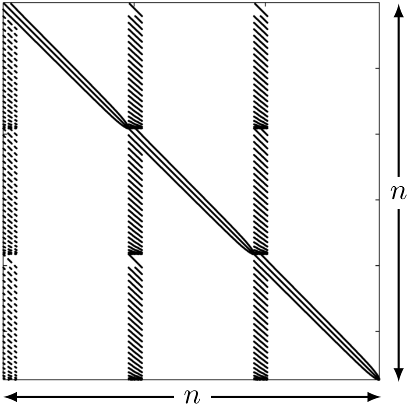

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Kelley, 2003

Path densities: Computation

1. Extract nutrient's regenerated source: $\mathbf{s}^{\mathsf{X}}_{\mathsf{reg}}(\mathbf{x})$ where e.g., $\mathsf{X} = \mathsf{Si}$.
2. Linear labelling/unlabelling equation: $(\partial_t + \mathbf{T} + \mathbf{L}_0)\mathbf{x}_{\mathsf{reg}} = \mathbf{s}^{\mathsf{X}}_{\mathsf{reg}}$
3. Use Green function to propagate from source on Ω_i:

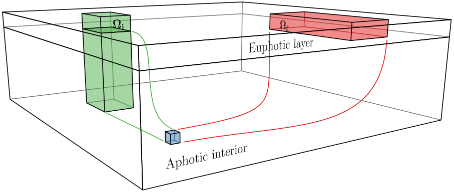
$$(\partial_t + \mathbf{T} + \mathbf{L}_0)\mathbf{g}_{\mathsf{reg}}(t) = \mathbf{0} \quad \text{and} \quad \mathbf{g}_{\mathsf{reg}}(0) = \mathrm{diag}(\mathbf{s}^{\mathsf{X}}_{\mathsf{reg}})\boldsymbol{\Omega}_i$$

4. Use Adjoint Green function to propagate to reemergence on Ω_f:

$$(-\partial_t + \tilde{\mathbf{T}} + \mathbf{L}_0)\tilde{\boldsymbol{\mathcal{G}}}_{\mathsf{reg}}(t) = \mathbf{0} \quad \text{and} \quad \tilde{\boldsymbol{\mathcal{G}}}_{\mathsf{reg}}(0) = \mathbf{V}\mathbf{L}_0\boldsymbol{\Omega}_f$$

5. Time integral by direct inversion:

$$\langle \mathbf{g}_{\mathsf{reg}} \rangle = (\mathbf{T} + \mathbf{L}_0)^{-1}\mathrm{diag}(\mathbf{s}^{\mathsf{X}}_{\mathsf{reg}})\boldsymbol{\Omega}_i$$
$$\langle \tilde{\boldsymbol{\mathcal{G}}}_{\mathsf{reg}} \rangle = (\tilde{\mathbf{T}} + \mathbf{L}_0)^{-1}\mathbf{V}\mathbf{L}_0\boldsymbol{\Omega}_f$$



6. Combine into path density:

$$\langle \boldsymbol{\eta}_{\mathsf{reg}}(\mathbf{r}) \rangle = \langle \tilde{\boldsymbol{\mathcal{G}}}_{\mathsf{reg}}(\mathbf{r}) \rangle \times \langle \mathbf{g}_{\mathsf{reg}}(\mathbf{r}) \rangle$$

(element-wise multiplication)